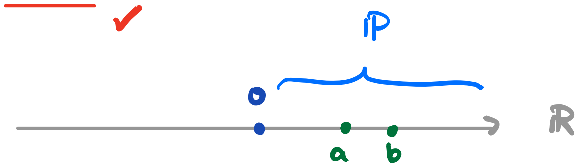
MATH 2050C Lecture 2 (Jan 13)

[[HW1 has been posted, due date Jan 21 (Fri).]]

Goal: $\mathbb{R}$ is a complete ordered field. ✓
today

Ordering $\rightsquigarrow$ $\mathbb{R}$ as a real line



Def^{n}/Thm: $\exists\ \phi \neq \mathbb{P} := \{$"positive" real numbers$\} \subseteq \mathbb{R}$ s.t.

(O1): $a, b \in \mathbb{P} \Rightarrow a+b,\ ab \in \mathbb{P}$

(O2): Trichotomy: $\forall\, a \in \mathbb{R}$, one and only one of the following holds:

$$a \in \mathbb{P} \quad \text{or} \quad a = 0 \quad \text{or} \quad -a \in \mathbb{P}$$

Notation: $a > 0$ if $a \in \mathbb{P}$; $a \geq 0$ if $a \in \mathbb{P}\ \{0\}$

$a < 0$ if $-a \in \mathbb{P}$; $a \leq 0$ if $-a \in \mathbb{P}\ \{0\}$

Define: $a > b$ if $a - b \in \mathbb{P}$

$a \geq b$ if $a - b \in \mathbb{P} \cup \{0\}$

Prop: (Rules of inequalities) Let $a, b, c \in \mathbb{R}$.

(a) $a > b$ and $b > c \Rightarrow a > c$

(b) $a > b \Rightarrow a + c > b + c$.

(c) $a > b \Rightarrow \begin{cases} ac > bc & \text{if } c > 0 \\ ac < bc & \text{if } c < 0 \end{cases}$

Proof: (a) By def^{n}, $a > b \Leftrightarrow a - b \in \mathbb{P}$

also $b > c \Leftrightarrow b - c \in \mathbb{P}$

By (O1), $a - c \underset{(A2),(A3),(A4)}{=} \underbrace{(a-b)}_{\in \mathbb{P}} + \underbrace{(b-c)}_{\in \mathbb{P}} \in \mathbb{P} \Rightarrow a > c$.

(b) Exercise.

(c) By def^n, $a > b \iff a - b \in \mathbb{P}$.

Given $c > 0$, ie. $c \in \mathbb{P}$, then by (O1)

$$ac - bc \overset{(D)}{=} \underset{\in \mathbb{P}}{(a-b)} \cdot \underset{\in \mathbb{P}}{c} \in \mathbb{P} \quad \Rightarrow \quad ac > bc.$$

Exercise for the case $c < 0$. $\square$

Thm 1: $\mathbb{P}$ contains all natural numbers, ie $\mathbb{N} \subset \mathbb{P}$

Lemma: $a^2 \geq 0 \quad \forall a \in \mathbb{R}$.

Proof: By (O2), there are 3 possible cases:

Case 1: $a \in \mathbb{P}$

$$a^2 = \underset{\in \mathbb{P}}{a} \cdot \underset{\in \mathbb{P}}{a} \overset{(O1)}{\in} \mathbb{P} \quad \text{so} \quad a^2 \geq 0.$$

Case 2: $a = 0$

$$a^2 = 0 \cdot 0 = 0 \quad \text{so} \quad a^2 \geq 0.$$

Case 3: $-a \in \mathbb{P}$

$$a^2 \overset{\text{Ex.}}{=} (-a)^2 = \underset{\in \mathbb{P}}{(-a)} \cdot \underset{\in \mathbb{P}}{(-a)} \overset{(O1)}{\in} \mathbb{P} \quad \text{so} \quad a^2 \geq 0.$$

$\square$

Proof of Thm 1: Use M.I. to show $n \in \mathbb{P} \quad \forall n \in \mathbb{N}$.

$n = 1$: $\quad 1 = 1 \cdot 1 = 1^2 \overset{\text{Lemma}}{\geq} 0$ and $1 \neq 0$ (by (M3)).

So, $1 \in \mathbb{P}$

Assume $n = k$ is true, ie. $k \in \mathbb{P}$.

Then $\underset{\in \mathbb{P}}{k} + \underset{\in \mathbb{P}}{1} \in \mathbb{P}$ by (O1), so $n = k+1$ is true. $\square$

Thm 2: $0 \leq a < \varepsilon \quad \forall \varepsilon > 0 \implies a = 0$.

(i.e. there is no "smallest" positive real number .)

Proof: By Contradiction. Suppose $a \neq 0$, then $a > 0$.

Note that $\frac{1}{2} > 0$ [Why? If not, then $-\frac{1}{2} \overset{(O2)}{>} 0$

$\implies \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = -1 \overset{(O1)}{>} 0$]

False $\because 1 > 0$

By (O1), $\underset{\mathbb{P}}{\frac{1}{2}} \cdot \underset{\mathbb{P}}{a} \in \mathbb{P}$, i.e. $\frac{1}{2}a > 0$.

Choose $\varepsilon = \frac{1}{2}a > 0$, by assumption. $a < \frac{1}{2}a$ False. (Ex: why?) $\square$

Prop: (1) $ab > 0 \implies$ either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

(2) $ab < 0 \implies$ either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

Pf: Exercise.